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ASYMPTOTIC EQUIVALENCE OF STATISTICAL INFERENCE BASED ON ALIGNED RANKS AND ON WITHIN-BLOCK RANKS

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1. Summary and introduction

Rank tests for the the null hypothesis of no treatment effect and rank-estimators of treatment effects, based on aligned ranks and on within-block ranks, are proposed for multiresponse experiments in two-way layouts without interaction, having one or more observations in each cell. Large sample properties of the tests and the estimators as cell sizes tend to infinity are investigated. It is shown that the aligned rank tests are asymptotically power-equivalent to the Friedman-type tests (within-block rank tests) and that the two R-estimators have asymptotically the same normal distribution. Further for the univariate case, it is found that the asymptotic relative efficiency (ARE) of the proposed rank test (R-estimator) with respect to the parametric F-test (parametric estimator) is equivalent to the classical ARE-result of the two-sample rank test with respect to the t-test and additionally asymptotically maximin power tests and minimax variance estimators due to Huber (1981) can be drawn.

For the two-way model, the k-th response $\mathbf{X}_{ijk}=(X_{ijk}^{(1)},\cdots,X_{ijk}^{(p)})'$ of the cell in the i-th block receiving the j-th treatment is expressed as

$$\mathbf{X}_{ijk}=\mu+\beta_i+\tau_j+e_{ijk} \quad (k=1,\cdots,n_j,\ j=1,\cdots,J,\ i=1,\cdots,I) \qquad (1.1)$$

where $\sum_{j=1}^{J} n_j\tau_j=0$, μ is the mean effect vector, β_i's are the block effects, τ_j's are the treatment effects and e_{ijk}'s are the independent error random vectors, each having identical continuous distribution function $F(\mathbf{x})$. The null hypothesis of interest and the alternative are respectively

H; $\tau_j=0$ for $j=1,\cdots,J$ and A; $\tau_j\neq 0$ for some j $(1\leq j\leq J)$.

Some rank test procedures for this model are already available. However, these procedures possess certain limitations. Friedman (1937) proposed a within-block rank test for designs having one observation per cell in the univariate case, that is, p=1. Mehra and Sarangi (1967) and Sen (1968) proposed aligned rank tests for p=1 independently, and Sen (1969) proposed aligned rank tests extended to multivariate case for designs having one observation per cell. All of them gave the asymptotic properties of their proposed tests as the number of blocks tends to infinity, that is, $I\to\infty$, and Mehra and Sarangi (1967) and Sen (1968) showed that the aligned rank tests are more efficient than the Friedman test in the sense of Pitman (1948).

On the other hand, Mack and Skillings (1980) proposed a univariate within-block rank test based on Wilcoxon score for the model having one or more observations per cell, including Friedman's test, gave the asymptotic distribution as cell sizes tend to infinity ($n_j \to \infty$), and investigated the asymptotic relative efficiency relative to the parametric F-test. But no-one has investigated the asymptotic properties of the aligned rank test statistics as the cell sizes tend to infinity for the model. We propose aligned rank tests and Friedman-type tests which are the extension of their proposed rank tests and investigate the asymptotic properties as the cell sizes tend to infinity. Sen and Puri (1977) proposed multivariate aligned rank tests for full rank linear models and investigated asymptotic properties of their proposed tests. However, the models do not include our model (1.1), which is not a full rank model.

Next we consider rank estimators of $\tau=(\tau_1,\cdots,\tau_J)$. For the model with one observation per cell, Puri and Sen (1967, 1971) proposed linear combinations of one-sample rank estimate statistics defined by Hodges and Lehmann (1963) as estimators of contrasts among $\tau_1,\cdots,\tau_J$ and gave the asymptotic variance as the number of blocks tends to infinity. We propose estimators of τ based on aligned ranks and on within-block ranks by straight method similar to the construction of Jureckova (1971), and investigate the asymptotic properties as the cell sizes tend to infinity.

In Section 2, linear rank statistics are introduced and, in Section 3, common assumptions for the asymptotic theory and basic theorems are given. In Section 4, the test procedures are proposed and the asymptotic distributions are drawn. In Section 5, the estimation theory is stated. In Section 6, the efficiencies and robustness are discussed.

2. Linear rank statistics

Let us define the aligned observations by

$\mathbf{Y}_{ijk}=\mathbf{X}_{ijk}-\hat{\beta}(\mathbf{X}_{i11},\cdots,\mathbf{X}_{iJn_J})$, where

$\hat{\beta}(\mathbf{X}_{i11},\cdots,\mathbf{X}_{iJn_J})=(\hat{\beta}_i^{(1)},\cdots,\hat{\beta}_i^{(p)})'$ and $\hat{\beta}_i^{(\ell)}=\hat{\beta}^{(\ell)}(X_{i11}^{(\ell)},\cdots,X_{iJn_J}^{(\ell)})$

is a translation equivariant symmetric function such that each aligned observation has a continuous distribution function. Then we can take, as $\hat{\beta}_i^{(\ell)}$, sample mean for $\{X_{ijk}^{(\ell)}: k=1,\cdots,n_j, j=1,\cdots,J\}$, sample median, trimmed mean, Winsorized mean, Hodges and Lehmann estimator, etc. Further for p-dimensional column vector $\mathbf{t}_j=(t_j^{(1)},\cdots,t_j^{(p)})'$ and J-dimensional row vector $\mathbf{t}^{(\ell)}=(t_1^{(\ell)},\cdots,t_J^{(\ell)})$, let $\mathbf{Y}_{ijk}(\mathbf{t}_j)=\mathbf{Y}_{ijk}-\mathbf{t}_j$ and $\mathbf{X}_{ijk}(\mathbf{t}_j)=\mathbf{X}_{ijk}-\mathbf{t}_j$ and let their ℓ-th coordinates be respectively

$Y_{ijk}^{(\ell)}(t^{(\ell)})=Y_{ijk}^{(\ell)}-t_j^{(\ell)}$ and $X_{ijk}^{(\ell)}(t^{(\ell)})$. Then putting $N=\sum_{j=1}^{J} n_j$ and $M=IN$, we define $R_{ijk}^{(\ell)}(t^{(\ell)})$ and $Q_{ijk}^{(\ell)}(t^{(\ell)})$ by the rank of $Y_{ijk}^{(\ell)}(t^{(\ell)})$ among the M observations $Y_{111}^{(\ell)}(t^{(\ell)}),\cdots,Y_{IJn_J}^{(\ell)}(t^{(\ell)})$ and the rank of $X_{ijk}^{(\ell)}(t^{(\ell)})$ among the N observations $X_{i11}^{(\ell)}(t^{(\ell)}),\cdots,X_{iJn_J}^{(\ell)}(t^{(\ell)})$ respectively. For the univariate case (p=1), $R_{ijk}^{(1)}(0)$'s and $Q_{ijk}^{(1)}(0)$'s are respectively aligned ranks defined by Mehra and Sarangi (1967) and within-block ranks defined by Friedman (1932). Using these ranks and score function $a_n^{(\ell)}(\cdot)$ which is a map from $\{1,\cdots,n\}$ to real values $(n\geq 1)$, for $t^{(\ell)}$, put

$$S_j^{(\ell)}(t^{(\ell)})=\sum_{i=1}^{I}\sum_{k=1}^{n_j}\{a_M^{(\ell)}(R_{ijk}^{(\ell)}(t^{(\ell)}))-\bar{a}_{Mi}^{(\ell)}(t^{(\ell)})\}/\sqrt{N} \tag{2.1}$$

and

$$T_j^{(\ell)}(t^{(\ell)})=\sum_{i=1}^{I}\sum_{k=1}^{n_j}\{a_N^{(\ell)}(Q_{ijk}^{(\ell)}(t^{(\ell)}))-\bar{a}_N^{(\ell)}\}/\sqrt{N}, \tag{2.2}$$

where $\bar{a}_{Mi}^{(\ell)}(t^{(\ell)})=\sum_{j=1}^{J}\sum_{k=1}^{n_j} a_M^{(\ell)}(R_{ijk}^{(\ell)}(t^{(\ell)}))/N$ and $\bar{a}_N^{(\ell)}=\sum_{m=1}^{N} a_N^{(\ell)}(m)/N$.

We define the random vectors with components of coordinates consisting of these simple linear rank statistics by

$$\mathbf{S}^{(\ell)}(\mathbf{t}^{(\ell)})=(S_1^{(\ell)}(\mathbf{t}^{(\ell)}),\cdots,S_J^{(\ell)}(\mathbf{t}^{(\ell)})), \tag{2.3}$$

$$\mathbf{T}^{(\ell)}(\mathbf{t}^{(\ell)})=(T_1^{(\ell)}(\mathbf{t}^{(\ell)}),\cdots,T_J^{(\ell)}(\mathbf{t}^{(\ell)})), \tag{2.4}$$

$\mathbf{S}(\mathbf{t})=(\mathbf{S}^{(1)}(\mathbf{t}^{(1)}),\cdots,\mathbf{S}^{(p)}(\mathbf{t}^{(p)}))'$ and

$\mathbf{T}(\mathbf{t})=(\mathbf{T}^{(1)}(\mathbf{t}^{(1)}),\cdots,\mathbf{T}^{(p)}(\mathbf{t}^{(p)}))'$ where $\mathbf{t}=(\mathbf{t}_1,\cdots,\mathbf{t}_J)$.

Since we consider the statistical inference based on $S_j^{(\ell)}(\mathbf{t}^{(\ell)})$'s and on $T_j^{(\ell)}(\mathbf{t}^{(\ell)})$'s and since the distributions of these statistics under the model (1.1) are independent of not only mean effect $\mathbf{u}$ but block effects $\boldsymbol{\beta}_i$'s, in the remainder of this paper, it is assumed without any loss of generality that

$$\mathbf{u}=\boldsymbol{\beta}_1=\cdots=\boldsymbol{\beta}_I=0. \tag{2.5}$$

Also for $\mathbf{b}^{(\ell)}=(b_1^{(\ell)},\cdots,b_I^{(\ell)})$ and $\mathbf{t}^{(\ell)}$, let $\tilde{R}_{ijk}^{(\ell)}(\mathbf{b}^{(\ell)},\mathbf{t}^{(\ell)})$ be the rank of $X_{ijk}^{(\ell)}(\mathbf{b}^{(\ell)},\mathbf{t}^{(\ell)})$ among the M observations $\{X_{ijk}^{(\ell)}(\mathbf{b}^{(\ell)},\mathbf{t}^{(\ell)});\ k=1,\cdots,n_j,\ j=1,\cdots,J,\ i=1,\cdots,I\}$, where $X_{ijk}^{(\ell)}(\mathbf{b}^{(\ell)},\mathbf{t}^{(\ell)})=X_{ijk}^{(\ell)}-b_i^{(\ell)}-t_j^{(\ell)}$. In order to investigate the asymptotic distributions of $S_j^{(\ell)}(\mathbf{t}^{(\ell)})$'s, we introduce the following statistic.

$$\tilde{S}_j^{(\ell)}(b^{(\ell)}, t^{(\ell)})$$

$$= \sum_{i=1}^{I} \sum_{k=1}^{n_j} \{a_M^{(\ell)}(\tilde{R}_{ijk}^{(\ell)}(b^{(\ell)}, t^{(\ell)})) - \bar{a}_{Mi}^{(\ell)}(b^{(\ell)}, t^{(\ell)})\}/\sqrt{N}, \qquad (2.6)$$

where $\bar{a}_{Mi}^{(\ell)}(b^{(\ell)}, t^{(\ell)}) = \sum_{j=1}^{J} \sum_{k=1}^{n_j} a_M^{(\ell)}(\tilde{R}_{ijk}^{(\ell)}(b^{(\ell)}, t^{(\ell)}))/N.$

To reduce notational comlexity, when t=0, we set $R_{ijk}^{(\ell)}=R_{ijk}^{(\ell)}(0)$, $Q_{ijk}^{(\ell)}=Q_{ijk}^{(\ell)}(0)$, $\bar{a}_{Mi}=\bar{a}_{Mi}(0)$, $S_j^{(\ell)}=S_j^{(\ell)}(0)$, $T_j^{(\ell)}=T_j^{(\ell)}(0)$, $S^{(\ell)}=S^{(\ell)}(0)$, $T^{(\ell)}=T^{(\ell)}(0)$, $S=S(0)$ and $T=T(0)$. Further when $b^{(\ell)}=0$ and $t^{(\ell)}=0$, we set $\tilde{R}_{ijk}^{(\ell)}=\tilde{R}_{ijk}^{(\ell)}(0,0)$ and $\tilde{S}_j^{(\ell)}=\tilde{S}_j^{(\ell)}(0,0)$.

Next we will investigate the moments of the rank statistics under H. So first we put matrix $Z=(Z_1, Z_2, \cdots, Z_I)$, where $Z_i=(Z_{i11}, \cdots, Z_{i1n_1}, Z_{i21}, \cdots, Z_{iJn_J})$ is a p×N matrix and let G_N be the finite group of translation $\{g_N\}$ such that $g_N(Z)=Z^*=(Z_1^*, \cdots, Z_I^*)$ and Z_i^* is any permutation of the columns of Z_i. G_N consists of $(N!)^I$ translations $\{g_N\}$. For any Z, we put $\mathcal{A}(Z)=\{g_N(Z); g_N \in G_N\}$. Let us now consider stochastic rank vectors $R_{ijk}=(R_{ijk}^{(1)}, \cdots, R_{ijk}^{(p)})'$ for $k=1, \cdots, n_j$, $j=1, \cdots, J$, $i=1, \cdots, I$ and put the collection rank matrix $R=(R_{111}, \cdots, R_{11n_1}, R_{121}, \cdots, R_{IJn_J})$.

Although the null distribution of R depends on $F(\mathbf{x})$ and is not distribution-free, the conditional distribution of R, given a set $\mathcal{A}(R^0)$, under H, is given by $P\{R=R^*|\mathcal{A}(R^0)\}=1/\{(N!)^I\}$ for all $R^*\in\mathcal{A}(R^0)$. (See Sen (1969)). Replacing $R_{ijk}^{(\ell)}$'s, R_{ijk}'s and R by $Q_{ijk}^{(\ell)}$'s, $\mathbf{Q}_{ijk}$'s and $\mathbf{Q}$ respectively, similarly we get the conditional null distribution $P\{\mathbf{Q}=\mathbf{Q}^*|\mathcal{A}(Q^0)\}=1/\{(N!)^I\}$ for all $\mathbf{Q}^*\in\mathcal{A}(\mathbf{Q}^0)$. From straightforward computations due to the above probabilities, it follows that the conditional expectations and conditional variance-covariance matrices of S and T under H are given by

$$E_0\{\mathbf{S}|\mathcal{A}(R)\}=E_0\{T|\mathcal{A}(Q)\}=0, \tag{2.7}$$

$$\text{Var-Cov}_0\{\mathbf{S}|\mathcal{A}(R)\}=\Gamma(R)\otimes\Lambda(n), \tag{2.8}$$

and

$$\text{Var-Cov}_0\{T|\mathcal{A}(Q)\}=\Gamma(Q)\otimes\Lambda(n), \tag{2.9}$$

where $\Gamma(R)=(\hat{\gamma}_{\ell\ell'}(R))_{\ell,\ell'=1,\cdots,p}$, $\Gamma(Q)=(\hat{\gamma}_{\ell\ell'}(Q))_{\ell,\ell'=1,\cdots,p}$,

$$\hat{\gamma}_{\ell\ell'}(R)=\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{n_j}\{a_M^{(\ell)}(R_{ijk}^{(\ell)})-\bar{a}_{Mi}^{(\ell)}\}\{a_M^{(\ell')}(R_{ijk}^{(\ell')})-\bar{a}_{Mi}^{(\ell')}\}/(N-1), \tag{2.10}$$

$$\hat{\gamma}_{\ell\ell'}(\mathbf{Q})=\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{n_j}\{a_N^{(\ell)}(\mathbf{Q}_{ijk}^{(\ell)})-\bar{a}_N^{(\ell)}\}\{a_N^{(\ell')}(\mathbf{Q}_{ijk}^{(\ell')})-\bar{a}_N^{(\ell')}\}/(N-1), \tag{2.11}$$

$$\Lambda(n)=((n_j/N)(\delta_{jj'}-n_{j'}/N))_{j,j'=1,\cdots,J}, \tag{2.12}$$

$\delta_{jj'}$ is the Kronecker delta and $A \otimes B$ denotes the Kronecker product of A and B.

3. Common assumptions and basic theorems

The following are the minimum assumptions to discuss the asymptotic theory.

Assumption 1. $\lim_{N\to\infty}(n_j/N)=\lambda_j>0$ for $j=1,\cdots,J$. □

Assumption 2. Score function $a_n^{(\ell)}(\cdot)$ is generated by a function $\Psi_\ell(u)$ $(0<u<1)$ by the following way $(\ell=1,\cdots,p)$:

$a_n^{(\ell)}(m)=E\{\Psi_\ell(U_n(m))\}$ or $\Psi_\ell(m/(n+1))$ for $m=1,\cdots,n$,

where $U_n(m)$ is the m-th order statistic in a sample of size n from the rectanguler (0,1) distribution. □

Assumption 3. The score generating function $\Psi_\ell(u)$ is non-constant, nondecreasing and square integrable. □

Assumption 4. For random variables $\hat{\beta}_i^{(\ell)}$'s which induce aligned observations, there exists some constant $\nu^{(\ell)}$ not depending on i such that $\hat{\beta}_i^{(\ell)}-\beta_i^{(\ell)}-\nu^{(\ell)}=O_p(1/\sqrt{N})$ $(i=1,\cdots,I,$ $\ell=1,\cdots,p)$, where as (2.5) is assumed, $\beta_i^{(\ell)}=0$. □

Assumption 5. Letting $F_\ell(x^{(\ell)})$ and $f_\ell(x^{(\ell)})$ be respectively the ℓ-th marginal distribution function of $F(\mathbf{x})$ and its density function, for $\ell=1,\cdots,p$, $F_\ell(x^{(\ell)})$ possess finite Fisher's information, i.e., $\int_{-\infty}^{\infty}\{-f_\ell'(x^{(\ell)})/f_\ell(x^{(\ell)})\}^2 f_\ell(x^{(\ell)})dx^{(\ell)}<+\infty$. □

Lemma 3.1. Let $(X_{111}^{(\ell)},\cdots,X_{IJn_J}^{(\ell)})$ have joint density $\prod_{i=1}^{I}\prod_{j=1}^{J}\prod_{k=1}^{n_j} f_\ell(x_{ijk}^{(\ell)})$ and let $\|z\|_m=\sqrt{z\cdot z'}$ for m-dimensional row vector z. Then under Assumptions 1 through 5, for any positive ε, C_1 and C_2,

$$\lim_{N\to\infty} P\{\sup_{\substack{\|\rho^{(\ell)}\|_I<C_1\\ \|\Delta^{(\ell)}\|_J<C_2}} |\tilde{S}_j^{(\ell)}(\rho^{(\ell)}/\sqrt{N},\Delta^{(\ell)}/\sqrt{N})-\tilde{S}_j^{(\ell)}+I\cdot d_\ell\cdot\Delta^{(\ell)}\sigma_j|>\varepsilon\}=0,$$

where $\tilde{S}_j^{(\ell)}(\rho^{(\ell)}/\sqrt{N},\Delta^{(\ell)}/\sqrt{N})$ is defined by (2.6),

$\rho^{(\ell)}=(\rho_1^{(\ell)},\cdots,\rho_I^{(\ell)})$, $\Delta^{(\ell)}=(\Delta_1^{(\ell)},\cdots,\Delta_J^{(\ell)})$,

$d_\ell=-\int_0^1\{\Psi_\ell(u)\cdot f_\ell'(F_\ell^{-1}(u))/f_\ell(F_\ell^{-1}(u))\}du$ and

$$\sigma_j=(-\lambda_j\lambda_1,\cdots,-\lambda_j\lambda_{j-1},\lambda_j-\lambda_j^2,-\lambda_j\lambda_{j+1},\cdots,-\lambda_j\lambda_J)'. \qquad (3.1)$$

□

Our main theorem of this Section is the following.

Theorem 3.2. Under the assumptions of Lemma 3.1, for any $\varepsilon>0$ and any $C>0$,

$$\lim_{N\to\infty} P\{\sup_{\|\Delta^{(\ell)}\|_J<C} |S_j^{(\ell)}(\Delta^{(\ell)}/\sqrt{N})-\tilde{S}_j^{(\ell)}+I\cdot d_\ell\cdot\Delta^{(\ell)}\cdot\sigma_j|>\varepsilon\}=0,$$

where $S_j^{(\ell)}(t^{(\ell)})$ is defined by (2.1). □

Let us define

$$E(C)=\{\Delta^{(\ell)};\Delta^{(\ell)}=(\Delta_1^{(\ell)},\Delta_2^{(\ell)},\cdots,\Delta_J^{(\ell)}),\ \|\Delta^{(\ell)}\|_J<C,\ \sum_{j=1}^{J} n_j\Delta_j^{(\ell)}=0\}.$$

Then we get two corollaries as direct results of Theorem 3.2.

Corollary 3.3. Under the assumptions of Lemma 3.1, for any $\varepsilon>0$ and any $C>0$,

$$\lim_{N\to\infty} P\{\sup_{\Delta^{(\ell)}\in E(C)} |S_j^{(\ell)}(\Delta^{(\ell)}/\sqrt{N})-\tilde{S}_j^{(\ell)}+I\cdot d_\ell\cdot\lambda_j\cdot\Delta_j^{(\ell)}|>\varepsilon\}=0. \quad \square$$

Corollary 3.4. Under the assumptions of Lemma 3.1, for any $\varepsilon>0$ and any $C>0$,

$$\lim_{N\to\infty} P\{\sup_{\Delta^{(\ell)}\in E(C)} ||S_j^{(\ell)}(\Delta^{(\ell)}/\sqrt{N})|-|\tilde{S}_j^{(\ell)}-I\cdot d_\ell\cdot\lambda_j\cdot\Delta_j^{(\ell)}||>\varepsilon\}=0. \quad \square$$

As in the proof of Lemma 3.1, we get

Theorem 3.5. Under the assumptions of Lemma 3.1, for any $\varepsilon>0$ and any $C>0$,

$$\lim_{N\to\infty} P\{\sup_{\|\Delta^{(\ell)}\|_J<C} |T_j^{(\ell)}(\Delta^{(\ell)}/\sqrt{N})-T_j^{(\ell)}+I\cdot d_\ell\cdot\Delta^{(\ell)}\cdot\sigma_j|>\varepsilon\}=0. \quad \square$$

4. Tests

Since the conditional expectations and conditional variance-covariance matrices of random vectors **S** and **T** are given by (2.7)-(2.12), we propose to test the null hypothesis H versus the alternative A, based on either of statistics

$$AL=\mathbf{S}'\{\Gamma(R)\otimes\Lambda(n)\}^{-}\mathbf{S} \text{ and } FR=\mathbf{T}'\{\Gamma(Q)\otimes\Lambda(n)\}^{-}\mathbf{T}.$$

Proposition 4.1. Suppose that $\Gamma(R)$ ($\Gamma(Q)$) is positive definite. Then AL (FR) does not depend on the choice of generalized inverse $\{\Gamma(R)\otimes\Lambda(n)\}^{-}$ ($\{\Gamma(Q)\otimes\Lambda(n)\}^{-}$) and is expressed as $AL=\mathbf{S}'\{\Gamma(R)^{-1}\otimes D(n)\}\mathbf{S}$ ($FR=\mathbf{T}'\{\Gamma(Q)^{-1}\otimes D(n)\}\mathbf{T}$), where $D(n)$ is the diagonal matrix with j-th diagonal element N/n_j. $\square$

For the univariate case, that is, $p=1$, Sen (1968) proposed the test based on AL, and Friedman (1937) proposed the test based on FR when $\Psi_1(u)=2u-1$ and $n_j=1$ for $j=1,\cdots,J$. Also Sen and Puri (1977) proposed aligned rank tests for limited multivariate linear models not including the model (1.1). Our test procedures are the extension of Sen (1968) and Friedman (1937).

First we will give the asymptotic convergences of the conditional variance-covariance matrices of **S** and T. Under Assumption 1, we get

$$\Lambda(n) \longrightarrow \Lambda=(\sigma_1,\cdots,\sigma_J) \text{ as } N\to\infty, \tag{4.1}$$

where σ_j's are defined by (3.1).

Here we set

Assumption 6. $\Psi_\ell(u)$'s are absolutely continuous. □

Let us put $\gamma_{\ell\ell'}=I\int_0^1\{\Psi_\ell(u)-\bar{\Psi}_\ell\}^2du$ if $\ell=\ell'$;

$=I\int_{R^2}\{\Psi_\ell(F_\ell(x))-\bar{\Psi}_\ell\}\{\Psi_{\ell'}(F_{\ell'}(y))-\bar{\Psi}_{\ell'}\}dF_{\ell\ell'}(x,y)$ otherwise, and

$\Gamma=(\gamma_{\ell\ell'})_{\ell,\ell'=1,\cdots,p}$, where $\bar{\Psi}_\ell=\int_0^1\Psi_\ell(u)du$ and $F_{\ell\ell'}(x,y)$ stands for the (ℓ,ℓ')-th marginal distribution of $F(\mathbf{x})$. Then we get

Lemma 4.2. Suppose that Assumptions 1 through 4 and Assumption 6 are satisfied. Then under H, both $\Gamma(R)$ and $\Gamma(Q)$ converge in probability to Γ. □

To derive asymptotic distributions of AL and FR, we set

Assumption 7. Γ is positive definite. □

Theorem 4.3. Suppose that Assumptions 1 through 7 are satisfied. Then under H, as $N\to\infty$, AL and FR have asymptotically the same χ^2-distribution with $p(J-1)$ degrees of freedom. □

Next we consider the sequence of local alternatives

$$A_N;\ \tau_j=\Delta_j/\sqrt{N},\ \Delta_j\neq\Delta_{j'},\ \text{for some } j\neq j' \text{ and } \sum_{j=1}^{J} n_j\Delta_j=0,$$

where $\Delta_j=(\Delta_j^{(1)},\cdots,\Delta_j^{(p)})'$.

If we suppose next Assumption 8, from the proof similar to the proof of Hajek and Sidak (1967), we find that A_N is contiguous to H as $N\to\infty$.

Assumption 8. $\int_{R^p}\{-\partial f(\mathbf{x})/\partial x^{(\ell)}/f(\mathbf{x})\}^2 f(\mathbf{x})d\mathbf{x}<\infty$ for $\ell=1,\cdots,p$ and $\partial f(\mathbf{x})/\partial x^{(\ell)}$'s are continuous. □

Theorem 4.4. Suppose that Assumptions 1 through 8 are satisfied. Then under A_N, as $N\to\infty$, AL and FR have asymptotically the same noncentral χ^2-distribution with $p(J-1)$ degrees of freedom and noncentrality parameter δ^2, where $\delta^2=\mu'(\Gamma\otimes\Lambda)^-\mu$, $\mu=(\mu_1^{(1)},\cdots,\mu_J^{(1)},\mu_1^{(2)},\cdots,\mu_J^{(p)})'$ and $\mu_j^{(\ell)}=I\cdot d_\ell\cdot\Delta^{(\ell)}\cdot\sigma_j$. □

5. Estimates

Using the method similar to Jureckova (1971), we propose the R-estimators of matrix τ, based on aligned ranks and on within-block ranks. Let $\|z\|=\sum_{j=1}^{J}|z_j|$ for J-dimensional row vector z. Then we put

$$D_N(R)=\{\theta: \sum_{\ell=1}^{p}\|S^{(\ell)}(\theta_1^{(\ell)},\cdots,\theta_J^{(\ell)})\|=\text{minimum under } \sum_{j=1}^{J}n_j\theta_j=0\}$$

$$=\{\theta:\|S^{(\ell)}(\theta_1^{(\ell)},\cdots,\theta_J^{(\ell)})\|=\text{minimum under } \sum_{j=1}^{J}n_j\theta_j^{(\ell)}=0 \text{ for } \ell=1,\cdots,p\}$$

and

$$D_N(Q)=\{\theta: \sum_{\ell=1}^{p}\|T^{(\ell)}(\theta_1^{(\ell)},\cdots,\theta_J^{(\ell)})\|=\text{minimum under } \sum_{j=1}^{J}n_j\theta_j=0\},$$

where $\theta=(\theta_1,\cdots,\theta_J)$, $\theta_j=(\theta_j^{(1)},\cdots,\theta_j^{(p)})'$ and $S^{(\ell)}(\cdot)$ and $T^{(\ell)}(\cdot)$ are defined by (2.3) and (2.4). Since $S^{(\ell)}(\cdot)$ and $T^{(\ell)}(\cdot)$ take finite values in $(\theta_1^{(\ell)},\cdots,\theta_J^{(\ell)})$, $D_N(R)$ and $D_N(Q)$ are not empty. We propose some point $\hat{\theta}_{AL}$ in $D_N(R)$ and some point $\hat{\theta}_{FR}$ in $D_N(Q)$ as an aligned rank estimator of τ and as a within-block rank estimator respectively. It is simple to verify

$$\{\theta: \sum_{\ell=1}^{p}\|S^{(\ell)}(\theta_1^{(\ell)}+\tau_1^{(\ell)},\cdots,\theta_J^{(\ell)}+\tau_J^{(\ell)})\|=\text{minimum under } \sum_{j=1}^{J}n_j\theta_j=0\}$$

$$=\{\theta-\tau;\theta\in D_N(R)\}$$

and

$$\{\theta: \sum_{\ell=1}^{p} \|T^{(\ell)}(\theta_1^{(\ell)}+\tau_1^{(\ell)},\cdots,\theta_J^{(\ell)}+\tau_J^{(\ell)})\|=\text{minimum under } \sum_{j=1}^{J} n_j\theta_j=0\}$$

$$=\{\theta-\tau;\theta\in D_N(Q)\}.$$

If $D_N(R)$ ($D_N(Q)$) is a convex set, a natural choice of $\hat{\theta}_{AL}$ ($\hat{\theta}_{FR}$) is the center of gravity of $D_N(R)$ ($D_N(Q)$). We add

Assumption 9. $d_\ell>0$ for $\ell=1,\cdots,p$. □

In many cases, using integration by parts yields

$d_\ell=\int_{-\infty}^{\infty}\psi_\ell'(F_\ell(x))\{f_\ell(x)\}^2dx$. Thus Assumption 9 is feasible.

Then even if $D_N(R)$ ($D_N(Q)$) is not convex, we can show

Theorem 5.1. Suppose that Assumptions 1-5 and 9 are satisfied. Then

$$\lim_{N\to\infty} P\{\sup_{\theta,\ \theta^*\in D_N(R)} \sqrt{N}\|\theta-\theta^*\|_{pJ}>\varepsilon\}=\lim_{N\to\infty} P\{\sup_{\theta,\ \theta^*\in D_N(Q)} \sqrt{N}\|\theta-\theta^*\|_{pJ}>\varepsilon\}$$

$$=0 \text{ for } \varepsilon>0,$$

where $\|A\|_{pJ}=\sqrt{\{\mathrm{vec}(A)\}'\cdot\{\mathrm{vec}(A)\}}$.

Furthermore $\sqrt{N}\cdot\mathrm{vec}(\hat{\theta}_{AL}-\tau)$ and $\sqrt{N}\cdot\mathrm{vec}(\hat{\theta}_{FR}-\tau)$ have the same multivariate normal distribution with mean 0 and variance-covariance matrix $\Xi\otimes\Lambda_0$, where vec(A) denotes $(a^{(1)},\cdots,a^{(p)})'$ for

$p \times J$ matrix $A=(a^{(1)'},\cdots,a^{(p)'})'$, $\Xi=(\xi_{\ell\ell'})_{\ell,\ell'=1,\cdots,p}$, $\xi_{\ell\ell'}=\gamma_{\ell\ell'}/(I^2 \cdot d_\ell \cdot d_{\ell'})$, $\Lambda_0=(\delta_{ij}/\lambda_i-1)_{i,j=1,\cdots,J}$ and δ_{ij} denotes the Kronecker delta. □

6. ARE and robustness in the case of p=1

At first, we investigate the asymptotic relative efficiencies (ARE's) of the proposed tests and estimators with respect to the normal theory parametric test and estimator. For $p \geqq 2$, the ARE's are complicated, especially in the case of tests, the ARE under A_N depends on parameter Δ and we can discuss the ARE as in Section 4 of Sen (1971). So we give the ARE's for p=1. It is simple to verify that (normalized likelihood ratio F-test) $\xrightarrow[\mathcal{L}]{}$ χ^2_{J-1} under H and $\xrightarrow[\mathcal{L}]{}$ $\chi^2_{J-1}(\eta^2)$ under A_N, where $\eta^2=I \cdot \Delta^{(1)}\Lambda^{-}\Delta^{(1)'}/\mathrm{Var}_0(X_{111})$. Also we can find that $\sqrt{N}(\bar{X}_{\cdot 1 \cdot}-\bar{X}_{\cdots},\cdots,\bar{X}_{\cdot J \cdot}-\bar{X}_{\cdots})' \xrightarrow[\mathcal{L}]{} N(0,\ \mathrm{Var}_0(X_{111})\cdot\Lambda_0/I)$, where $\bar{X}_{\cdot j \cdot}=\sum_{i=1}^{I}\sum_{k=1}^{n_j}X_{ijk}/(I \cdot n_j)$ and $\bar{X}_{\cdots}=\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{n_j}X_{ijk}/M$. Combining these facts with Theorems 4.3, 4.4 and 5.1, we get

ARE(AL, F-test)

$=\mathrm{ARE}(\hat{\theta}_{AL}$, parametric estimator)

$=\mathrm{Var}_0(X_{111})\cdot[\int_0^1 \Psi_1(u)\{-f'(F^{-1}(u))/f(F^{-1}(u))\}du]^2/\int_0^1 \{\Psi_1(u)-\bar{\Psi}_1\}^2 du$,

which is equivalent to the classical ARE-result of the two-sample rank test with respect to the t-test, where ARE(C, D) stands for the asymptotic relative efficiency of C with respect to D. We say that, when $V_N \xrightarrow[\mathcal{L}]{} N(0,\Sigma)$ and $W_N \xrightarrow[\mathcal{L}]{} N(0,c\Sigma)$ for c>1 (c<1), the asymptotic variance of V_N is smaller (larger) than that of W_N.

The larger $[\int_0^1 \Psi_1(u)\cdot\{-f'(F^{-1}(u))/f(F^{-1}(u))\}du]^2/\int_0^1 \{\Psi_1(u)-\bar{\Psi}_1\}^2 du$ is, the larger the asymptotic local power of the rank test is for fixed $\Delta^{(1)}$, and the smaller the asymptotic variance of the rank estimator is. Thus asymptotic optimal score generating function is given by $\Psi_1(u)=-f'(F^{-1}(u))/f(F^{-1}(u))$ for fixed f(x). Further the choice of $\Psi_1(u)$ which gives maximin asymptotic power of the rank test over the class of distributions that f(x) is in a contamination neighborhood and minimax asymptotic variance of the rank estimator is reviewed by Section 2.9 of Hettmansperger (1984) and Section 4.6 of Huber (1981). Also the choice of $\Psi_1(u)$ over the class that f(x) is in a Kolmogorov neighborhood is stated in Huber (1964) and Wiens (1986).

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